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Prof. Dr. Eliezer Martins Diniz

Universidade de São Paulo
FEA Ribeirão Preto

Universidade de São Paulo
Faculdade de Economia, Administração e Contabilidade de Ribeirão Preto

Faculdade de Economia, Administração e Contabilidade
de Ribeirão Preto -USP
Avenida dos BandeirantTes,3900
14049-900 Ribeirão Preto – SP

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A NOTE ON THE SOLOW GROWTH MODEL[1]

Eliezer M. Diniz*

Abstract

This paper presents an alternative diagrammatical exposition of the Solow model that explicit the connections with the more sophisticated treatment of the Ramsey-Cass-Koopmans model on intertemporal consumer choice. As simple as it is, the present treatment is not found in any paper or textbook as far as we know. The purpose of the paper is essentially to share with a greater audience a novel way to teach an old topic. We also discuss the conditions under which a Solow model may be viewed as a reduced form of the Ramsey-Cass-Koopmans model using the solution of the inverse optimum problem.

Resumo

Este trabalho apresenta uma exposição gráfica alternativa do modelo de Solow que deixa clara as conexões com o tratamento mais sofisticado do modelo de Ramsey-Cass-Koopmans sobre escolha intertemporal de consumo. O presente tratamento, apesar de simples, não é encontrado em nenhum artigo ou livro-texto do qual tenhamos conhecimento. O objetivo do trabalho é principalmente repartir com um público mais amplo uma nova forma de lecionar um tópico antigo. Também discutimos as condições sob as quais um modelo de Solow pode ser visto como uma forma reduzida do modelo de Ramsey-Cass-Koopmans utilizando para esse fim a solução do problema do ótimo inverso.

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* Professor of Economics - Universidade de São Paulo, Faculdade de Economia, Administração e Contabilidade - Av. dos Bandeirantes 3900, 14040-900, Ribeirão Preto, SP, Brazil - e-mail: elmdiniz@usp.br

1. Introduction

The Solow growth model is nowadays a standard tool to introduce economic growth to undergraduate students. Some of them express anxiety when the Ramsey-Cass-Koopmans model is presented in advanced courses. This occurs more frequently when we teach development economics, as it is necessary to teach both models one after another. The reason is that they do not understand the connections between both models and, by implication, are not able to evaluate benefits and limitations in each case.

Our purpose in this paper is to share with a greater audience an educational strategy we have used during the past years in undergraduate classes. It makes a natural transition from the Solow model to the Ramsey-Cass-Koopmans model by means of a new diagram. As simple as it is, it is impressive that no one adopted it either in papers or in textbooks. The discussion is completed with the conditions to consider the Solow model as a reduced form of the Ramsey-Cass-Koopmans model based on the solution of the inverse optimal problem.

2. The Solow growth model

The following exposition uses standard assumptions. Suppose that the economy is closed, we have a large number of inhabitants, and the same is also true for firms. There is no government and no taxes whatsoever. The income in terms of the single good may be used either for consumption or investment, like the "farm animals" interpretation in Barro and Sala-i-Martin (1995). This gives the usual identity:

$$Y = C + I$$

where *Y*, *C* and *I* are income, consumption and gross investment. Output is obtained using three inputs: capital (*K*), labour (*L*) and labour-augmenting technology (*A*). The composed variable *AL* is sometimes called "effective labour". It is assumed that labour and technology grow at the constant exogenous rates *n* and *g*, respectively. The production function $Y = F(K, AL)$ satisfies standard properties, namely

PF1: $F(0, AL) = 0$, $F(K,0) = 0$ (each input is essential to production).

PF2: $F_K(K, AL) > 0$, $F_L(K, AL) > 0$, $F_{KK}(K, AL) < 0$, $F_{LL}(K, AL) < 0$, $F_{KL}(K, AL) > 0$ (the marginal product of any input is positive and decreasing).[2]

PF3: $\lim_{K \to 0} F_K(K, AL) = \infty$, $\lim_{L \to 0} F_L(K, AL) = \infty$, $\lim_{K \to \infty} F_K(K, AL) = 0$, $\lim_{L \to \infty} F_L(K, AL) = 0$ (the relative scarcity of an input has a direct relationship with its marginal product).

PF4: $\lambda Y = F(\lambda K, \lambda AL)$ (the production function is homogeneous of degree one).

Assume also that capital depreciates at the fixed rate δ. Call $\dot{K}$ net investment. The previous assumptions give the usual net investment equation:

$$\dot{K} = F(K, AL) - C - \delta K .$$

The same equation can be rewritten in intensive form (every variable divided by effective labour).[3] This gives the well-known equation on the dynamics of the net investment per worker:

$$\dot{k} = f(k) - c - (n + g + \delta)k \qquad (1)$$

[2] Remember that $F_L = AF_{AL}$ so that the same properties hold for both F_L and F_{AL}. Similar remarks hold for the other partial derivatives on *L*.

[3] Sometimes it is assumed that a fixed proportion of the population works. This assumption does not fit the data well, as the labour force participation rate is not the same around the world. The abnormal observations come from some Asian countries (for example, Japan and China). *Cf.* Jones (2002), ch. 1.

where $f(\cdot)$ is the intensive form of the production function with the following properties implied by PF1-PF4:

IPF: $f(0) = 0,\ \lim_{k \to 0} f'(k) = \infty,\ \lim_{k \to \infty} f'(k) = 0.$

This set of properties is known in the economic literature as the Inada conditions.

The expression (1) is the key equation in an economic growth model. It shows the dynamic behaviour of the state variable (capital per worker). The $\dot{k} = 0$ curve is used to find the steady state equilibrium. The results are presented in Figure 1 and in the following expression

$$c^* = f(k^*) - (n + g + \delta)k^*. \tag{2}$$

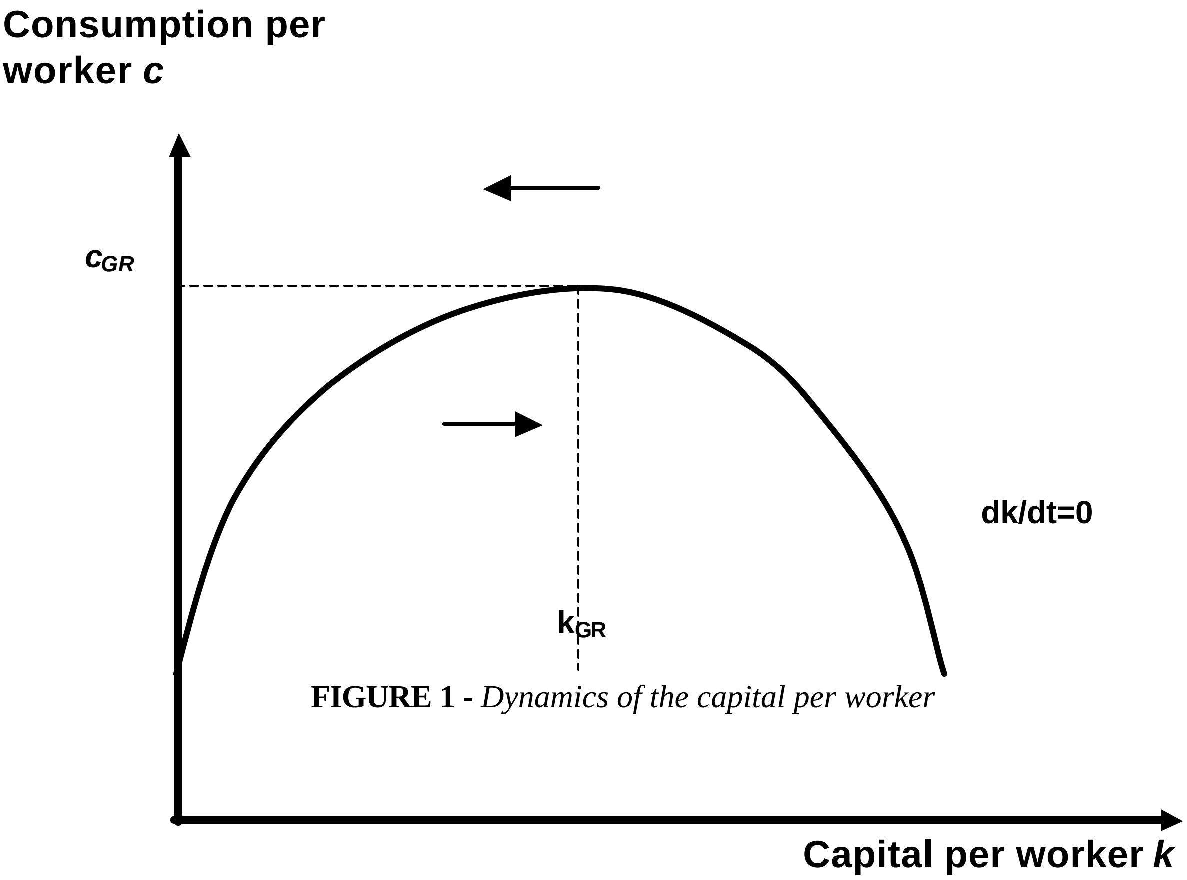

FIGURE 1 - *Dynamics of the capital per worker*

The representation of (2) in Figure 1 is the easiest way to identify the golden rule stock of capital per worker (k_{GR}) and its associated maximum consumption (c_{GR}).

The distinctive feature of the Solow model is the particular form of the consumption function:

$$c = (1 - s) f(k) \tag{3}$$

that assumes a constant marginal propensity to consume out of income. This function is like a rule of thumb that dictates the behaviour of economic agents. An alternative way to interpret this curve is to call it an *iso-saving rate locus*, so that in every point one has the same propensity to save.

Substitute (3) for consumption per worker in the dynamics equation (1). The result is the typical dynamics equation of the Solow model as detailed in Solow (1956):

$$\dot{k} = sf(k) - (n + g + \delta)k,$$

which usually appears in textbooks in a figure with two curves ("actual investment" and "break-even investment") and an equilibrium at the point they intersect.

It is proposed in this paper to study the Solow model in a novel way. The dynamics can be found if we trace (2) and (3), obtaining Figure 2. Its main lesson is that the economy is always on the consumption curve. In fact, the adjustment path is coincident with the consumption curve. If one is below (above) the $\dot{k} = 0$ curve, then capital and consumption increases (decreases) over time on the path given by the consumption function. The steady state equilibrium is given by point E, where both curves intersect.

Figure 2 also shows additional useful information: total output per worker associated with the equilibrium capital stock per worker; real wage; rental rate of capital (namely real interest rate plus depreciation rate). The last two results follow naturally from the Euler Theorem, as the production function is homogeneous of degree one. It is possible to analyse the qualitative effects of any change in the economic system on real wage and real interest rate behaviour over time (since the depreciation rate is constant). It is also possible to divide income into parcels: consumption and investment, or real wages and capital rent. It is a powerful diagram more useful than the traditional presentation of the Solow model. The connections between the figure and the Ramsey-Cass-Koopmans will become apparent in the sequel.

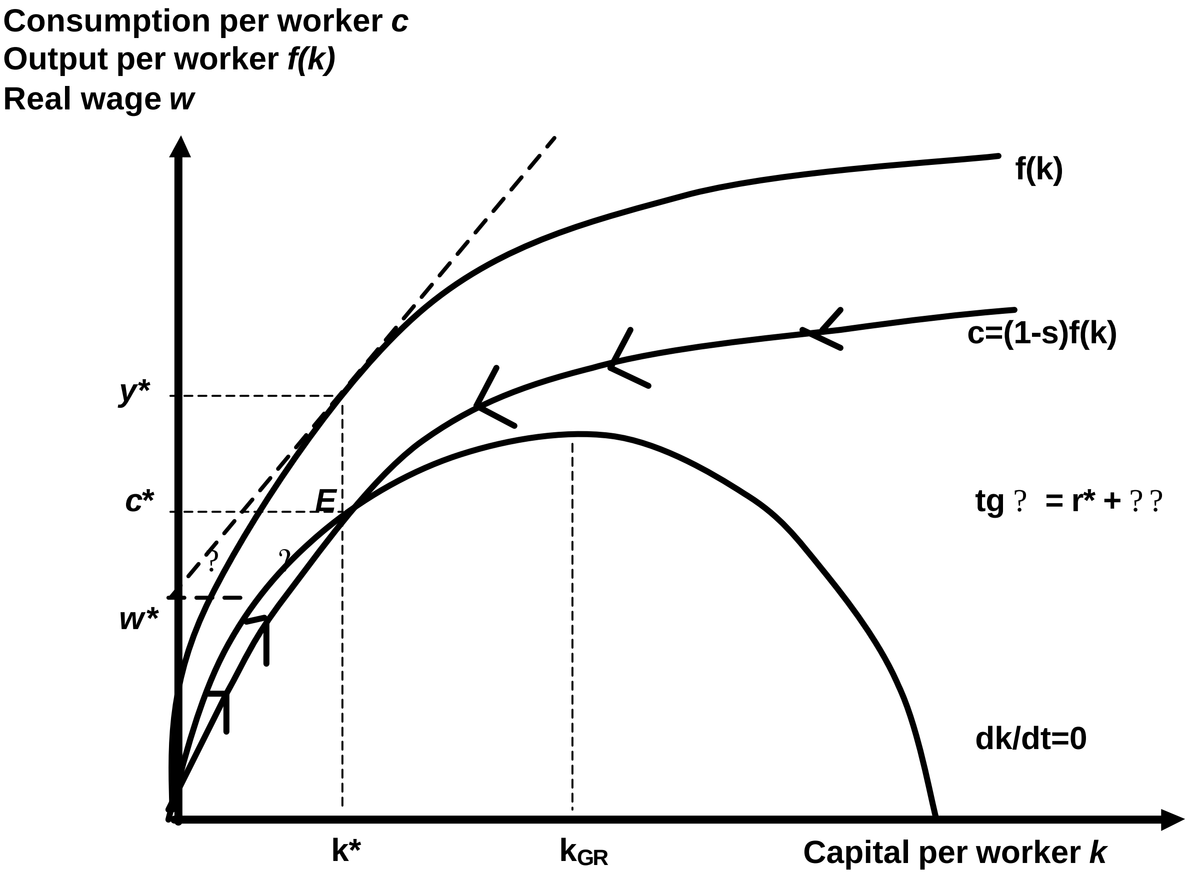

FIGURE 2 - *Dynamics of the Solow growth model*

A technical detail that deserves explanation is the position of the curves $f(k)$, $(1 - s) f(k)$ and $\dot{k} = 0$ near the origin of the plane. The derivatives of the three curves are respectively given by

$$\frac{dy}{dk} = f'(k)\,, \quad \frac{dc}{dk} = (1 - s) f'(k)\,, \quad \left.\frac{dc}{dk}\right|_{\dot{k}=0} = f'(k) - (n + g + \delta)\,.$$

It is easy to see that

$$\frac{dy}{dk} ? \frac{dc}{dk} \text{ and } \frac{dy}{dk} ? \left.\frac{dc}{dk}\right|_{\dot{k} ? 0}, \; ? k, k ? 0 .$$

The above inequalities provide a foundation for the result that the production function is always above both the consumption function (as expected) and the locus of equilibrium stock of capital per worker. The relationship between the consumption function and the locus of equilibrium stock of capital per worker is more ingenious. The essential point is to compare $sf'(k)$ with $(n ? g ? ?)$. If $k ? 0$ then clearly $sf'(k) ? (n ? g ? ?)$ using the Inada conditions. This implies

$$\frac{dc}{dk} ? \left.\frac{dc}{dk}\right|_{\dot{k} ? 0} .$$

If $k ? k_{GR}$ then clearly $sf'(k) ? (n ? g ? ?)$ as we have $f'(k) ? (n ? g ? ?)$ by the definition of golden rule. This implies

$$\frac{dc}{dk} ? \left.\frac{dc}{dk}\right|_{\dot{k} ? 0} .$$

Both results and the Inada conditions imply the form we traced both curves in Figure 2.

Using this diagram it is easier to see the impact of a change in the propensity to save on consumption. If the propensity to save out of income is higher, the iso-saving rate curve shifts downwards and to the right, and the new equilibrium has a higher consumption-capital pair. This result is clearer than the usual derivations trying to show under what conditions the impact on consumption is positive or negative.[4]

The adjustment of the Solow model to real world data is better the smaller the variance of transitory income compared to the variance of the permanent income. This is what produce empirical estimates showing that consumption is proportional to income. If Friedman is right, then Solow is also right in choosing that consumption function. But Solow took this short cut instead of choosing consumption from a utility maximising analysis with a feeling that this would produce a tight approximation to real world movements without the burden of a more complex theory. Or, using Solow's own words: "The art of succesful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive."[5] The Ramsey-Cass-Koopmans class of models took exactly this starting point to go on further into the analysis of economic growth. But it is necessary to dig deeper in order to understand some subtleties of the argument and to have a fair evaluation of both models.

3. The Ramsey-Cass-Koopmans model[6]

In this section we show the main results of the RCK model and then discuss briefly its relationship with the Solow model.

The RCK model uses a utility maximising framework not present in the Solow analysis, so saving is endogenous. Total utility as of time 0 (U_0) is given by

$$U_0 ? \int_0^{?} u(c_t) \exp(??t) dt \qquad (4)$$

[4] See, for instance, Romer (2001), pp. 17-22.
[5] Solow (1956), p. 65.
[6] RCK model henceforth.

where $u(c_t)$ is the instantaneous utility at time t and $? ? 0$ is the rate of time preference. The problem here is to maximise (4) subject to (1), nonnegativity constraints on consumption and capital, and an initial condition on the stock of capital per worker. This utility-maximising exercise produces a result that is well-known to growth theorists, namely the dynamics of consumption per worker:

$$?_c ? ? (c_t)[f'(k_t) ? (n ? g ? ? ? ?)]$$

where $?_c$ is the rate of growth of consumption per worker and $?(c_t)$ is the instantaneous elasticity of substitution. Using a general form of utility function as we did in (4), it is easy to show that $?(c_t)$ depends on the utility function. The consumption per worker equilibrium locus $(?_c ? 0)$ paralell to the vertical axis of Figure 1 is given by the modified golden rule

$$f'(k^*) ? (n ? g ? ? ? ?) \tag{5}$$

where $k^* ? k_{GR}$. Expression (5) gives the optimal stock of capital per worker independent of the level of the consumption per worker. The interpretation of this result is the following: if the population growth rate, or the technology growth rate, or the depreciation rate, or the rate of time preference, is higher (lower), then the optimal stock of capital per worker is lower (higher). Population growth and technology progress exerts an impact into the denominator of the variables, so the intuition for the above result is clear; a higher depreciation means that capital scraps faster; and the higher the rate of time preference, the greater is the impatience over future consumption, so there will be less saving and less capital per worker in equilibrium. The iso-saving rate locus helps to reveal that each point on the equilibrium locus $?_c ? 0$ is related to a particular propensity to save: a lower level of consumption for a given equilibrium stock of capital per worker is associated with a higher propensity to save.

In addition to (5), the intertemporal maximisation also must satisfy (1) and the transversality condition

$$\lim_{t ? ?} ?(t)k(t) ? 0 \tag{6}$$

where $?(t)$ is the shadow price of capital in terms of utility. The transversality condition is sometimes neglected in less technical discussions, and the consequence is that it is commonly misunderstood. This condition is useful to obtain a policy function that gives the consumption-capital pair on the trajectory to the steady state equilibrium (the saddle point path in the RCK model). In some cases it is possible to have an analytical resolution of the policy function. This is a point we explore further in the next section.

A comparison of between models reveal that the consumption function of the Solow model always intercepts the consumption equilibrium locus of the RCK model only once. The question we need to address is the following: Is this point of special significance or not? Answer: There is nothing that ties the equilibrium point of the iso-saving locus of the Solow model to the equilibrium point of the RCK model. The adjustment trajectory of the former has no relation to the corresponding one of the latter. This fact may be interpreted in the following sense: The Solow model uses a consumption function that is consistent with real world data. The RCK model shows the behaviour of the optimal consumption that maximises utility. Both equilibria are not necessarily equal. A possible reason is that economic agents are not utility maximisers, prefering to use simple rules of thumb in day by day decisions.

We need to ask under what conditions the equilibrium of the Solow model is the same of the RCK model. It is possible to derive the conditions under which the Solow equilibrium is the same as the RCK equilibrium with almost no cost in terms of modelling.

Let us start with the RCK equilibrium. The steady state is characterised by (2), (5) and the transversality condition (6). Equations (2) and (5) imply

$$c^* = f(k^*) - [f'(k^*) - \rho]k^* . \qquad (7)$$

Dividing both sides by output and rearranging we obtain the propensity to save out of income as a function of the steady state sock of capital per worker:

$$s_{RCK} = s(k^*) = \left(1 - \frac{\rho}{f'(k^*)}\right) Sh(k^*) \qquad (8)$$

where $Sh(k^*)$ is the elasticity of output per worker with respect to capital per worker (equal to the share of income per worker that goes to capital owners). In the steady state we have a stable share of capital, in accordance with the collection of empirical facts on economic growth.[7] The expression (8) has an economic meaning if and only if $f'(k^*) > \rho$, which imply that $0 < s_{RCK} < 1$. This condition is satisfied if and only if the effective depreciation $(n + g + \delta)$ is positive (see (5)). As this inequality always occurs, (8) has an economic meaning.

The Solow equilibrium is characterised by (2) and (3). It will also be a utility maximising equilibrium if and only if $s_{SOLOW} = s_{RCK}$. In this case, the steady state stock of capital per worker is given by the modified golden rule. The implied level of consumption $c = [f(k) - kf'(k)] + \rho k$ also comes from the modified golden rule. Even if the equilibrium is the same, the adjustment path will not necessarily be the same. This is a point we want to address in the next section.

4. The inverse optimal problem in the Solow model

The hidden face of the Solow model is that the consumption function assumed throughout the analysis reflects the dynamics of the adjustment to the steady state. It is like the saddle point path in the RCK model. Every time we estimate a consumption function, we have a picture of the saddle point path. The Solow model is valid as a reduced form of the RCK model if and only if the Solowian consumption function is equal to to the saddle point path of the larger model. Putting in other words, if and only if the Solowian consumption function is equal to the policy function obtained by means of the transversality condition in the RCK model.

We need to address the following question: Under what conditions the utility function generates a saddle point path equal to the postulated Solowian consumption function? This is known in the literature as the inverse optimum problem and was addressed masterfully by Chang (1988). Assume an economic agent faces the same decision problem as posed for the RCK model. Assume further a policy function $c = (1 - s) f(k)$. If the policy function was generated by a utility function $u(c)$ with a discount rate ρ then the following first order differential equation need to be solved:

$$u''(c) + \varphi(k)u'(c) = 0 \qquad (9)$$

[7] *Cf.* for example Jones (2002), ch. 1, Fact #5.

where $?(k)\ ?\ \frac{[f'(k)\ ?\ (n\ ?\ g\ ?\ ?\ ?\ ?)]}{[sf(k)\ ?\ (n\ ?\ g\ ?\ ?)k](1\ ?\ s)f''(k)}$, given two initial conditions $u'(c_0)$ and $u(c_0)$.

A solution for marginal utility is given by

$$u'(c)\ ?\ u'(c_0)\exp\left\{ ?\int_{c_0}^{c} ?\left[f^{?1}\left(\frac{x}{1\ ?\ s} \right) \right] dx \right\} \tag{10}$$

using the initial condition $u'(c_0)\ ?\ 0$. The exponential form of the solution ensures that the marginal utility is always positive ($u'(c)\ ?\ 0$) if we assume $?(?)\ ?\ 0$ by means of $k_0\ ?\ k^*$ and the sufficient condition $f'(k)\ ?\ (n\ ?\ g\ ?\ ?)$. Using all assumptions in (9) we obtain $u''(c)\ ?\ 0$. From (10) we obtain a solution for the utility function

$$u(c)\ ?\ u(c_0)\ ?\int_{c_0}^{c} u'(c_0)\exp\left\{ ?\int_{c_0}^{m} ?\left[f^{?1}\left(\frac{x}{1\ ?\ s} \right) \right] dx \right\} dm \tag{11}$$

given the initial conditions $u(c_0)\ ?\ 0$ and $u'(c_0)\ ?\ 0$. The whole family of solutions for utility functions in (11) produce a policy function given by the Solowian consumption function. In other words, the Solow model is a reduced form of the RCK model if the utility function satisfies (11).

5. Summary

This paper was intended to present the well-known Solow model in a novel way. We reviewed the Solow growth model in order to show the connections between this framework and the RCK growth model. A new diagram of the Solow model proved to be more useful than the common textbook treatment in order to introduce ideas and to make a link with the RCK model. It was also useful to show clearly the effect of a change in the saving rate on the consumption path. As far as we know there is no similar diagram in papers or textbooks to date. In addition we discussed that the Solow model can be viewed as a reduced form of the RCK model if and only if the Solowian consumption function is coincident with the saddle point path of the larger model. The conditions for this relationship to hold are developed in the last section of the paper and are based on the solution of the inverse optimal problem.

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